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Extremal self-dual $[40, 20, 8]$ codes with covering radius 7

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Abstract

We construct new extremal self-dual $[40, 20, 8]$ codes with covering radius 7. It is also shown that the vectors of a fixed weight in a coset of weight $4n+2$ in an extremal doubly even self-dual code of length $24n+16$ such that the coset has no vector of weight $4n+4$ form a 1-design.

Keywords: Extremal self-dual code; Covering radius; Harmonic weight enumerator and neighbor

1. Introduction

Let C be a binary self-dual code of length n, that is, C is an $n/2$-dimensional vector subspace of $\mathbb{F}_2^n$ with $C = C^\perp$ where $C^\perp$ is the dual code under the standard inner product, namely, $C^\perp := \{x \in \mathbb{F}_2^n \mid x \cdot y = 0 \text{ for all } y \in C\}$. A self-dual code such that all codewords have weights divisible by four is called doubly even. Such codes exist if and only if $n \equiv 0 \pmod 8$. Self-dual codes that are not doubly even are called singly

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even. The minimum weight d of a self-dual code of length n is upper bounded by $d \leqslant 4[n/24]+6$ if $n \equiv 22 \pmod{24}$ and $d \leqslant 4[n/24]+4$ otherwise (see [11,16]). A self-dual code meeting the bound is called extremal.

The covering radius $R(C)$ of a code C is the smallest integer R such that spheres of radius R around codewords of C cover the space $\mathbb{F}_2^n$. The covering radius is a basic and important geometric parameter of a code. A vector a of a coset $U := x + C$ is called a coset leader of U if the weight of a is minimal in U and the weight of a coset U is defined as the weight of a coset leader. The covering radius is the same as the largest weight of all the coset leaders of the code (cf. [1,3]). The covering radii of extremal self-dual codes have been investigated in [1,9]. By the sphere-covering bound and the Delsarte bound (cf. [1]), the covering radius of an extremal doubly even self-dual code of length 40 is between 6 and 8 [1]. It is worthwhile to describe the current status of the existence problem of such a code with covering radius 6. In [9,14], based on a preprint by Michio Ozeki, the non-existence of an extremal doubly even self-dual $[40,20,8]$ code with covering radius 6 was announced. However, unfortunately, his preprint contained an error and, in his paper [15] he withdrew the above announcement. An extremal doubly even self-dual code of length 40 with covering radius 7 was discovered in [9], namely D_{40} in [9]. This is the only known extremal doubly even self-dual code with covering radius not meeting the Delsarte bound. A large number of extremal doubly even self-dual codes for length 40 exist (cf. [10]). This is one reason of interest in extremal doubly even self-dual codes with covering radius 6 or 7.

In this note, a new extremal doubly even self-dual $[40,20,8]$ code with covering radius 7 is constructed. This is found by considering doubly even self-dual neighbors of extremal double circulant doubly even self-dual codes given in [8]. Similarly we also have found extremal singly even $[40,20,8]$ codes with covering radius 7 as singly even self-dual neighbors. It is mentioned in [9] that there are six cosets of weight 6 in D_{40} which have no vector of weight 8 and the set of vectors of the coset of any given weight $i (6 \leqslant i \leqslant 34)$ forms a 1-design. It is shown in this note that a coset of weight $4n+2$ in an extremal doubly even self-dual code of length $24n+16$ such that the coset has no vector of weight $4n+4$ has the same property.

2. A new extremal doubly even code with covering radius 7

Assmus and Pless [1] gave various bounds on the covering radius. They also determined the covering radii of extremal doubly even self-dual codes of length $n \leqslant 32$ and 48, and gave bounds on the covering radii of all other extremal doubly even self-dual codes of lengths up to 96. Let C be any extremal doubly even self-dual code of length 40. As described in Section 1, $6 \leqslant R(C) \leqslant 8$. In this section, we demonstrate the following:

Proposition 1. *There are at least two inequivalent extremal doubly even self-dual* $[40,20,8]$ *codes with covering radius 7.*

Recall that self-dual codes C, C' of length n are *neighbors* if the dimension of $C \cap C'$ is $n/2 - 1$. The method to find all self-dual neighbors for a given self-dual code has been recently developed by the second author [13]. This method was used to classify all singly even self-dual codes of length 32. Double circulant codes are divided into two types, namely, the pure type and the bordered type. There are exactly three inequivalent extremal bordered double circulant doubly even self-dual codes of length 40, namely $C_{40,1}$, $C_{40,2}$, $C_{40,3}$ and there are exactly nine inequivalent extremal bordered double circulant doubly even self-dual codes, namely $C_{40,4}, \dots, C_{40,12}$ in [8] while $C_{40,3}$ and $C_{40,5}$ are equivalent. Hence there are exactly 11 inequivalent extremal double circulant doubly even self-dual odes of length 40 [8].

By the method in [13], we have found all extremal doubly even self-dual neighbors of the 11 double circulant codes. This calculation was done by Magma. Then only one extremal doubly even self-dual code C_{40} with covering radius 7 is found. This code is a neighbor of $C_{40,6}$. Its generator matrix is given by (I, M). To save space M is written as $m_1, m_2, \dots, m_{20}$ where m_i denotes the ith row of M:

01110001001000001010, 11110110111000010100, 11000001010111111010,
01100000101011111101, 10110000010101111110, 00101100011101011110,
11011000010010111110, 11010110000010101111, 11101011000001010111,
11110101100000101011, 10001110100111110100, 10001001001111101011,
01111110101100000101, 10111111010110000010, 01011111101011000001,
10101111110101100000, 01010111111010110000, 00101011111101011000,
00010101111110101100, 00001010111111010110.

Our code C_{40} is not equivalent to the known code D_{40} in [9]. Indeed, we have verified that C_{40} is generated by the codewords of minimum weight, however, the codewords of the minimum weight in D_{40} generate a subcode of dimension 17. Hence the two codes are inequivalent. Recall that an automorphism of C is a permutation of the coordinates of C which preserves C, and the automorphism group of C is the set of all automorphisms of C. Note that D_{40} in [9] has the automorphism group of order 44236800. The automorphism group of C_{40} is of order 6144 and its orbits are

$$\begin{aligned} o_1 &:= \{4, 9, 14, 19, 23, 28, 33, 38\}, \\ o_2 &:= \{1, 2, 6, 7, 11, 12, 16, 17, 21, 25, 26, 30, 31, 35, 36, 40\}, \\ o_3 &:= \{3, 5, 8, 10, 13, 15, 18, 20, 22, 24, 27, 29, 32, 34, 37, 39\}. \end{aligned}$$

The code obtained from C_{40} by shortening the two orbits o_2, o_3 is the code generated by the all-one vector. The two codes obtained by shortening two orbits o_1, o_2 and o_1, o_3 are doubly even self-dual codes of length 16. It was shown by Magma that both codes are equivalent to the direct sum of two copies of the extended Hamming $[8, 4, 4]$ code. The code obtained by shortening the orbit o_2 (resp. o_3) is a doubly even $[24, 8, 8]$ code with weight enumerator $1 + 71(y^8 + y^{16}) + 112y^{12} + y^{24}$ (resp. $1 + 39(y^8 + y^{16}) + 176y^{12} + y^{24}$). The code obtained by shortening the orbit o_1 is a $[32, 13, 8]$ code with weight enumerator $1 + 140(y^8 + y^{24}) + 1472(y^{12} + y^{20}) +$

$4966y^{16} + y^{32}$. It is an interesting problem to characterize C_{40} by considering shortened codes.

The possible coset weight distributions of an extremal doubly even self-dual code of length 40 were determined in [14]. Using these, we have calculated the complete coset weight distributions of C_{40}. In order to save space, the distributions are listed using a different style from usual ones. We give the possible coset weight enumerators W_i of cosets of weight i using parameters $t_{i,j}$:

$$\begin{aligned} W_1 \coloneqq {} & y + 57y^7 + 228y^9 + 6384y^{11} + 14896y^{13} + 95988y^{15} + 143982y^{17} \\ & + 262752y^{19} + \cdots \end{aligned}$$

$$\begin{aligned} W_2(t_2) \coloneqq {} & y^2 + t_2y^6 + (-2t_2 + 114)y^8 + (-5t_2 + 2033)y^{10} + (9044 + 12t_2)y^{12} \\ & + (9t_2 + 47139)y^{14} + (118446 - 30t_2)y^{16} + (-5t_2 + 212971)y^{18} \\ & + (40t_2 + 269080)y^{20} + \cdots \end{aligned}$$

$$\begin{aligned} W_3(t_3) \coloneqq {} & y^3 + t_3y^5 + (3t_3 + 21)y^7 + (-19t_3 + 619)y^9 + (4125 - t_3)y^{11} \\ & + (77t_3 + 23247)y^{13} + (-41t_3 + 77073)y^{15} + (-135t_3 + 168111)y^{17} \\ & + (251091 + 115t_3)y^{19} + \cdots \end{aligned}$$

$$\begin{aligned} W_4(t_{4,1}, t_{4,2}) \coloneqq {} & t_{4,1}y^4 + t_{4,2}y^6 + (160 + 8t_{4,1} - 2t_{4,2})y^8 + (1664 - 5t_{4,2} - 64t_{4,1})y^{10} \\ & + (60t_{4,1} + 10560 + 12t_{4,2})y^{12} + (192t_{4,1} + 9t_{4,2} + 44160)y^{14} \\ & + (120160 - 30t_{4,2} - 328t_{4,1})y^{16} + (216320 - 5t_{4,2} - 128t_{4,1})y^{18} \\ & + (262528 + 40t_{4,2} + 518t_{4,1})y^{20} + \cdots \end{aligned}$$

$$\begin{aligned} W_5(t_5) \coloneqq {} & t_5y^5 + (32 + 3t_5)y^7 + (-19t_5 + 544)y^9 + (4416 - t_5)y^{11} \\ & + (77t_5 + 22848)y^{13} + (-41t_5 + 76768)y^{15} + (-135t_5 + 169440)y^{17} \\ & + (250240 + 115t_5)y^{19} + \cdots \end{aligned}$$

$$\begin{aligned} W_6(t_6) \coloneqq {} & t_6y^6 + (-2t_6 + 160)y^8 + (-5t_6 + 1664)y^{10} + (12t_6 + 10560)y^{12} \\ & + (9t_6 + 44160)y^{14} + (-30t_6 + 120160)y^{16} + (-5t_6 + 216320)y^{18} \\ & + (40t_6 + 262528)y^{20} + \cdots \end{aligned}$$

Table 1
The numbers of the different cosets

W_1									total
	#								40
$W_2(t_2)$	t_2	5	11	17	23	29	41		total
	#	272	408	60	24	8	8		780
$W_3(t_3)$	t_3	0	1	2	3	4	5	7	total
	#	2304	3392	2368	736	448	296	336	9880
$W_4(t_{4,1}, t_{4,2})$	$(t_{4,1}, t_{4,2})$	$(1,0)$	$(1,2)$	$(1,4)$	$(1,6)$	$(1,8)$	$(1,10)$	$(1,12)$	
	#	1728	16096	24256	14944	10016	2624	2928	
	$(t_{4,1}, t_{4,2})$	$(1,14)$	$(1,16)$	$(1,18)$	$(1,22)$	$(1,24)$	$(2,0)$	$(2,2)$	
	#	384	640	576	32	96	144	1344	
	$(t_{4,1}, t_{4,2})$	$(2,4)$	$(2,6)$	$(2,8)$	$(2,10)$	$(2,12)$	$(2,14)$	$(2,16)$	
	#	1584	768	1044	992	552	984	14	
	$(t_{4,1}, t_{4,2})$	$(2,18)$	$(2,22)$	$(2,24)$	$(3,0)$	$(3,2)$	$(3,4)$	$(3,6)$	
	#	72	16	12	32	32	192	32	
	$(t_{4,1}, t_{4,2})$	$(3,8)$	$(3,10)$	$(4,8)$	$(4,10)$	$(4,16)$	$(5,4)$	$(6,0)$	
	#	96	96	78	32	1	16	1	
	$(t_{4,1}, t_{4,2})$	$(8,0)$							total
	#	6							82460
$W_5(t_5)$	t_5	1	2	3	4	5	6	8	total
	#	269504	110992	31552	9688	2048	1088	48	424920
$W_6(t_6)$	t_6	2	4	6	8	10	12	14	
	#	14016	60320	122928	120790	69024	31576	14936	
	t_6	16	18	20	22	24	26	30	
	#	4209	1600	1168	64	288	96	24	
	t_6	36							total
	#	8							441047
W_7									total
	#								89448

$$W_7 := 32y^7 + 544y^9 + 4416y^{11} + 22848y^{13} + 76768y^{15} + 169440y^{17} + 250240y^{19} + \cdots.$$

We list the numbers of the cosets with the possible coset weight enumerators in Table 1.

3. Cosets of extremal doubly even codes of length $24n + 16$ and 1-designs

In [9], an interesting property of D_{40} was observed, namely, there are six cosets of weight 6 which have no vector of weight 8, and the set of vectors of the coset of any given weight $i (6 \leqslant i \leqslant 34)$ forms a 1-design. The purpose of this section is to explain this property as follows:

Theorem 2. *Let C be an extremal doubly even self-dual code of length $24n+16$ and U be a coset of C of weight $4n+2$. If the coset U has no vector of weight $4n+4$, then the set of vectors of U of any given weight $i(4n+2 \leqslant i \leqslant 20n+14)$ forms a 1-design.*

Let N be a positive integer and let X_k be the set of all k-subsets of $\{1,2,\dots,N\}$ for $0 \leqslant k \leqslant N$.

We recall the definition and properties of designs. Let t,i,λ be positive integers with $t \leqslant i$. Let $\mathscr{B} \subset X_i$. We say $\mathscr{B}$ is a t-(N,i,λ) design (or t-design simply) if $|\{U \in \mathscr{B} \mid T \subset U\}| = \lambda$ for all $T \in X_t$.

Set $V := \mathbb{F}_2^N$. We denote by $\mathbb{R}X_k$ and $\mathbb{R}V$ the free real vector spaces spanned by the elements of X_k and V, respectively. An element of $\mathbb{R}X_k$ is denoted by

$$f = \sum_{a \in X_k} f(a)a.$$

We identify an element of V with its support. An element $f \in \mathbb{R}X_k$ can be extended to an element of $\tilde{f} \in \mathbb{R}V$ by setting, for all $u \in V$,

$$\tilde{f}(u) := \sum_{\substack{z \in X_k \\ z \subset u}} f(z).$$

The *differentiation* γ is the operator on $\mathbb{R}V$ defined by linearity from

$$\gamma(z) := \begin{cases} \sum_{y \in X_{k-1},\ y \subset z} y & \text{for } z \in X_k \quad \text{and for } k = 1,2,\dots,N, \\ 0 & \text{for } z \in X_0, \end{cases}$$

and the harmonic space Harm_k is the kernel of γ:

$$\mathrm{Harm}_k := \mathrm{Ker}(\gamma|_{\mathbb{R}X_k}),$$

for all $k = 1,2,\dots,N$ and $\mathrm{Harm}_0 := \mathbb{R}$. The following lemma gives the characterization of designs in terms of the harmonic spaces.

Lemma 3 (Delsarte [6]). *Let i,t be integers such that $0 \leqslant t \leqslant i \leqslant N$. A subset $\mathscr{B} \subset X_i$ is a t-design if and only if $\sum_{b \in \mathscr{B}} \tilde{f}(b) = 0$, for all $f \in \mathrm{Harm}_k,\ 1 \leqslant k \leqslant t$.*

Now we introduce the harmonic weight enumerator which is defined by Bachoc [2]. Let U be a subset of V and $f \in \mathrm{Harm}_k$. The harmonic weight enumerator associated to U and f is

$$W_{U,f}(x,y) := \sum_{u \in U} \tilde{f}(u) x^{N-\mathrm{wt}(u)} y^{\mathrm{wt}(u)}.$$

For a nonnegative integer i and $U \subset V$, define $A_i(U) := |\{u \in U \mid \mathrm{wt}(u) = i\}|$ and $W_U(x,y) := W_{U,1}(x,y) = \sum_{i=0}^{N} A_i(U) x^{N-i} y^i$.

Lemma 4. *Let C be a code of length N, $a \in V$, and $f \in \mathrm{Harm}_k$.*

$$\sum_{v \in C^\perp} (-1)^{v \cdot a} \tilde{f}(v) x^{N-\mathrm{wt}(v)-k} y^{\mathrm{wt}(v)-k} = \frac{(-2)^k}{|C|} \sum_{u \in a+C} \tilde{f}(u)(x+y)^{N-\mathrm{wt}(u)-k}(x-y)^{\mathrm{wt}(u)-k}.$$

Proof. Similar to the proof of Theorem 2.1 in [2]. □

The *Krawtchouk polynomial* is defined to be

$$P_i(x; N) := \sum_{j=0}^{i} (-1)^j \binom{N-x}{i-j} \binom{x}{j}.$$

Let C be a binary linear code. It is shown in [5] that $\mathrm{wt}(U) \leqslant s'(C) := |\{\mathrm{wt}(c) \mid 0 \neq c \in C^\perp\}|$ for any coset U of C where $\mathrm{wt}(U)$ is the weight of U. Let $\mathscr{S} := \{\sigma_1, \ldots, \sigma_{s''}\}$ be a set of nonzero integers such that $\{\mathrm{wt}(u) \mid 0 \neq u \in C^\perp\} \subset \{\sigma_1, \ldots, \sigma_{s''}\}$. An *annihilator polynomial* of C is defined to be

$$\alpha_{\mathscr{S}}(x) := \frac{2^N}{|C|} \prod_{i=1}^{s''} \left(1 - \frac{x}{\sigma_i}\right).$$

Expand $\alpha_{\mathscr{S}}(x)$ in terms of $P_i(x; N)$:

$$\alpha_{\mathscr{S}}(x) = \sum_{i=0}^{s''} \alpha_{\mathscr{S},i} P_i(x; N). \tag{1}$$

Lemma 5 (MacWilliams et al. [12] Chapter 6, Theorem 20). *For a coset U of a code C, $W_U(x, y)$ is uniquely determined by $A_0(U), \ldots, A_{s'(C)-1}(U)$, and* $\sum_{i=0}^{s''} \alpha_{\mathscr{S},i} A_i(U) = 1$.

For the remainder of this section, $N := 24n + 16$, C denotes an extremal doubly even self-dual code of length N and U is a coset of C with weight $4n + 2$ which has no vector of weight $4n + 4$.

Lemma 6. *For any $b \in X_1$, $\mathrm{wt}(b + U) = 4n + 1$, and the coset weight enumerator of $b + U$ is uniquely determined and does not depend on the choice of b.*

Proof. Set $\mathcal{S} := \{4n+4i \mid 1 \leqslant i \leqslant 4n+3\} \cup \{24n+16\}$. Note that $\{\mathrm{wt}(u) \mid 0 \neq u \in C\} \subset \mathcal{S}$. From

$$\begin{aligned}\alpha_{\mathcal{S}}(x) =& \frac{2^{24n+16}}{|C|}\left(1-\frac{x}{24n+16}\right)\prod_{i=1}^{4n+3}\left(1-\frac{x}{4n+4i}\right)\\ =& \frac{n!2^{4n-1}}{(3n+2)(5n+3)!}\Bigg(x^{4n+4}-4(4n+5)(3n+2)x^{4n+3}\\ &+\frac{16(4n+3)(54n^3+151n^2+129n+35)}{3}x^{4n+2}\\ &-\frac{64(3n+2)(2n+1)(4n+3)(36n^3+107n^2+93n+25)}{3}x^{4n+1}+\cdots\Bigg),\end{aligned}$$

and

$$\begin{aligned}P_i(x;N) :=& \sum_{r=0}^{i}(-1)^r\binom{N-x}{i-r}\binom{x}{r}\\ =& \frac{(-1)^i 2^i}{i!}x^i+\frac{(-1)^{i+1}2^{i-1}N}{(i-1)!}x^{i-1}+\frac{(-1)^i 2^{i-3}(2i-4+3N^2-3N)}{(i-2)!3}x^{i-2}\\ &+\frac{(-1)^{i+1}2^{i-4}N(2i-4+N^2-3N)}{(i-3)!3}x^{i-3}+\cdots,\end{aligned}$$

we have

$$\begin{aligned}\alpha_{\mathcal{S},4n+4} &= \frac{(4n+4)!n!}{2^5(5n+3)!(3n+2)}, \alpha_{\mathcal{S},4n+3} = \frac{(4n+3)!n!}{2^2(5n+3)!},\\ \alpha_{\mathcal{S},4n+2} &= \frac{(4n+4)!n!}{2^4(5n+3)!(3n+2)}, \alpha_{\mathcal{S},4n+1} = \frac{-3(4n+3)!n!}{2^2(5n+3)!}\end{aligned}$$

in the formula (1). We have

$$A_{4n+2}(U) = -\frac{A_{4n+4}(U)}{2}+2^4(3n+2)\frac{(5n+3)!}{(4n+4)!n!}$$

from Lemma 5. Note that $\mathrm{wt}(b+u)=\mathrm{wt}(u)-1$ or $\mathrm{wt}(u)+1$ for all $u \in U$. So $\mathrm{wt}(b+U)=4n+1$ or $4n+3$. As $A_{4n+4}(U)=0$,

$$2^4(3n+2)\frac{(5n+3)!}{(4n+4)!n!} = A_{4n+2}(U) = A_{4n+1}(b+U)+A_{4n+3}(b+U).$$

From Lemma 5,

$$\begin{aligned}1 &= \alpha_{\mathcal{S},4n+1}A_{4n+1}(b+U)+\alpha_{\mathcal{S},4n+3}A_{4n+3}(b+U)\\ &= \frac{(4n+3)!n!}{2^2(5n+3)!}(-3A_{4n+1}(b+U)+A_{4n+3}(b+U)).\end{aligned}$$

We have

$$A_{4n+1}(b+U) = \frac{4(2n+1)(5n+3)!}{(4n+4)!n!}, A_{4n+3}(b+U) = \frac{4(10n+7)(5n+3)!}{(4n+4)!n!}.$$

So $\mathrm{wt}(b+U) = 4n+1$ and $W_{b+U}(x,y)$ is uniquely determined from Lemma 5 and does not depend on b since $s'(C) \leqslant 4n+4 = |\mathscr{S}|$. $\square$

We now give a proof of Theorem 2. Note that Harm_1 is spanned by $b' - b$ for all $b, b' \in X_1$. Theorem 2 holds if we show $W_{f,U}(x,y) = 0$ for all $f = 2(b'-b) \in \mathrm{Harm}_1$ from Lemma 3. Let $a \in U : U = a + C$. Note that $\tilde{f}(v) = (-1)^{b \cdot v} - (-1)^{b' \cdot v}$ and

$$\begin{aligned}(-1)^{a \cdot v}\tilde{f}(v) &= (-1)^{a \cdot v}((-1)^{b \cdot v} - (-1)^{b' \cdot v}) \\ &= (-1)^{(a+b)\cdot v} - (-1)^{(a+b')\cdot v}. \end{aligned} \tag{2}$$

From Lemma 4 for $1 \in \mathrm{Harm}_0$,

$$\sum_{v \in C} (-1)^{v\cdot(a+b)} x^{N-\mathrm{wt}(v)} y^{\mathrm{wt}(v)} = \frac{1}{|C|} \sum_{u \in a+b+C} (x+y)^{N-\mathrm{wt}(u)} (x-y)^{\mathrm{wt}(u)}. \tag{3}$$

As the RHS of formula (3) does not depend on the choice of b from Lemma 6, the LHS of the formula has the same property. Hence

$$\begin{aligned} 0 &= \sum_{v \in C} (-1)^{v\cdot(a+b)} x^{N-\mathrm{wt}(v)-1} y^{\mathrm{wt}(v)-1} - \sum_{v \in C} (-1)^{v\cdot(a+b')} x^{N-\mathrm{wt}(v)-1} y^{\mathrm{wt}(v)-1} \\ &= \sum_{v \in C} (-1)^{v \cdot a} \tilde{f}(v) x^{N-\mathrm{wt}(v)-1} y^{\mathrm{wt}(v)-1} \quad \text{by (2)}, \\ &= \frac{-2}{|C|} \sum_{u \in a+C} \tilde{f}(u)(x+y)^{N-\mathrm{wt}(u)-1}(x-y)^{\mathrm{wt}(u)-1}. \end{aligned}$$

for all $f = 2(b'-b) \in \mathrm{Harm}_1$ from Lemma 4. We have the desired results. $\square$

Let d_{16}^+ be the (extremal) doubly even self-dual code of length 16 with generator matrix $(I_8,\ J_8 - I_8)$, where I_8 is the identity matrix and J_8 is the all-one matrix of order 8. The code d_{16}^+ also has exactly one coset which satisfies the conditions in Theorem 2. However, we have not succeeded in finding more extremal doubly even $[40, 20, 8]$ codes with the coset satisfying the condition.

4. New extremal singly even codes with covering radius 7

4.1. An alternative construction of S_{40}

In [9], an extremal singly even self-dual $[40, 20, 8]$ code with covering radius 7 is also constructed. We denote this code by S_{40}. It is easy to see that the code S_{40} is a neighbor of D_{40}. In this section, we give an alternative construction of S_{40} by relating the extended Golay [24, 12, 8] code.

The following construction method is well-known as the $|u\,|\,u+v|$ construction. Let C_1 and C_2 be codes with parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$, respectively. Then

$$(C_1, C_2) := \{(u, u+v) | u \in C_1, v \in C_2\}$$

is a $[2n, k_1 + k_2, \min\{2d_1, d_2\}]$ code. Let G_{20} be the self-orthogonal [20, 8, 8] code which is the shortened code from the binary extended Golay code by deleting any four coordinates. Note that if C is self-orthogonal then $(C^{\perp}, C)$ is self-dual. In addition, we have verified that $(G_{20}^{\perp}, G_{20})$ is equivalent to S_{40}. It is worthwhile to mention that there is a unique [20, 8, 8] code up to equivalence [7].

4.2. Singly even codes with covering radius 7

We also consider all extremal singly even self-dual [40, 20, 8] neighbors with covering radius 7 of all the 11 extremal double circulant doubly even self-dual codes given in [8] and our code C_{40} discovered in Section 2.

All extremal singly even self-dual neighbors with covering radius 7 are listed in Table 2. The second and sixth columns list the extremal doubly even self-dual code C under consideration. The third and seventh columns give the orders of their automorphism groups. The fourth and eighth columns give the value β in their weight enumerators. We note that the possible weight enumerator of an extremal singly even self-dual code of length 40 is given in [4] by

$$1 + (125 + 16\beta)y^8 + (1664 - 64\beta)y^{10} + \cdots \quad .$$

The code $N_{40,13}$ is the same as the singly even code S_{40} found in [9]. Generator matrices of the codes $N_{40,i}$ $(i \neq 13)$ in the table are given in Appendix. We also give the numbers n_i of distinct cosets of weight i. The numbers $n_i (4 \leqslant i \leqslant 7)$ are listed in Table 3 noting that $n_0 = 1,\ n_1 = 40,\ n_2 = 780$ and $n_3 = 9880$ for every code.

Recently Ozeki [15] gave the complete coset weight distribution of some extremal singly even self-dual [40, 20, 8] code with covering radius 7 and weight enumerator $\beta = 10$ however he did not describe how to construct the code. We have verified that the code is the same as S_{40}. Indeed, from the form of the generator matrix given in [15, p. 556], the code is a neighbor of D_{40}.

Appendix

To save space, we only list matrices $M_{40,i}$ in generator matrices $(I, M_{40,i})$ in standard form for $N_{40,i}$ $(i \neq 13)$. For $N_{40,i}$ $(i \neq 1, 6, 9, 13, 19, 20)$, the matrices define actual neighbors without coordinate permutations. The matrices are written

Table 2
Extremal singly even self-dual $[40, 20, 8]$ neighbors

Codes	C	|Aut|	β	Codes	C	|Aut|	β
$N_{40,1}$	$C_{40,1}$	3	1	$N_{40,2}$	$C_{40,2}$	1	1
$N_{40,3}$	$C_{40,2}$	1	2	$N_{40,4}$	$C_{40,2}$	2	1
$N_{40,5}$	$C_{40,4}$	8	2	$N_{40,6}$	$C_{40,4}$	1	1
$N_{40,7}$	$C_{40,4}$	4	3	$N_{40,8}$	$C_{40,4}$	2	1
$N_{40,9}$	$C_{40,7}$	2	1	$N_{40,10}$	$C_{40,9}$	2	1
$N_{40,11}$	$C_{40,12}$	1728	1	$N_{40,12}$	$C_{40,12}$	98304	2
$N_{40,13}$	$C_{40,12}$	44236800	10	$N_{40,14}$	C_{40}	4	1
$N_{40,15}$	C_{40}	256	2	$N_{40,16}$	C_{40}	32	1
$N_{40,17}$	C_{40}	64	2	$N_{40,18}$	C_{40}	192	3
$N_{40,19}$	C_{40}	256	2	$N_{40,20}$	C_{40}	1024	8
$N_{40,21}$	C_{40}	6144	6				

Table 3
The numbers of distinct cosets of weight i

Codes	n_4	n_5	n_6	n_7
$N_{40,1}$	86617	387152	436890	127216
$N_{40,2}$	86693	390297	436814	124071
$N_{40,3}$	86205	386434	437302	127934
$N_{40,4}$	86715	390996	436792	123372
$N_{40,5}$	86153	384692	437354	129676
$N_{40,6}$	86674	389270	436833	125098
$N_{40,7}$	85771	384312	437736	130056
$N_{40,8}$	86703	390194	436804	124174
$N_{40,9}$	86685	389658	436822	124710
$N_{40,10}$	86651	388580	436856	125788
$N_{40,11}$	87210	406008	436297	108360
$N_{40,12}$	86663	400888	436844	113480
$N_{40,14}$	86740	392621	436767	121747
$N_{40,15}$	86248	388480	437259	125888
$N_{40,16}$	86762	393136	436745	121232
$N_{40,17}$	86214	389344	437293	125024
$N_{40,18}$	85768	386894	437739	127474
$N_{40,19}$	86248	388352	437259	126016
$N_{40,20}$	83468	372760	440039	141608
$N_{40,21}$	84476	386840	439031	127528

in octal using $0 = (000)$, $1 = (001), \ldots, 6 = (110)$ and $7 = (111)$, where $a = (1)$ and $b = (0)$.

$$
\begin{aligned}
N_{40,1} = {} & 41625310444124614651522743220666264133536726413615 \\
& 53127432511263756563770637624116567465202716335740 \\
& 352050671745556044473422462706771a,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,2} = {} & 3777777624131246224670117637011763372453766632 5470 \\
& 117747011774701177470117747011774701137470123 74701 \\
& 637470123747022374704237471023747b,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,3} = {} & 377777746620321664227011763376416670117647011 77470 \\
& 117747011774701177470116630350774701137470123 74701 \\
& 637470123747022374704237471023747b,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,4} = {} & 377777737022437524746717155460431670117650077 41711 \\
& 450333320063767217470117747011604162701524663 74701 \\
& 747213715613762374704237471023747b,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,5} = {} & 410465700066170006617000660700066274104607000 67070 \\
& 006707000630700063274107307000330700033070003 30700 \\
& 423274003307000330700033070003307b,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,6} = {} & 415255743361150677747002712333420676610646543 27443 \\
& 076501711311262357340705360115125123330377003 30700 \\
& 426444001543400154340015434033446a,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,7} = {} & 570015436131117435002706122700066204330750067 25070 \\
& 006707000630700060665701444361640174305127043 30700 \\
& 543604335250040330700033070003307b,
\end{aligned}
$$

$$
\begin{aligned}
N_{40,8} = {} & 254174153105222632067000660700066156340122471 31070 \\
& 006621346130700076366333307000164015660113351 53367 \\
& 033070050264352504170033070003307b,
\end{aligned}
$$

$$\begin{aligned}N_{40,9} = {} & 656502715116654063156723011270455572762623510621224\\ & 5724555456703047067302365333004573664535563706702\\ & 117273022675354475354047535421266 1a,\end{aligned}$$

$$\begin{aligned}N_{40,10} = {} & 452417102765453424433027655270262230276524503537 42\\ & 771055506161626412630277263027353141065460514431 15\\ & 707473420033753726301372630137263b,\end{aligned}$$

$$\begin{aligned}N_{40,11} = {} & 12372113612213664021247642130623573062357306234730\\ & 62367306236730623673062167306216730631673061167306\\ & 116730711673071167303116730311673b,\end{aligned}$$

$$\begin{aligned}N_{40,12} = {} & 05234270023667062356306235730623573062357306234730\\ & 62367306236730623613052173300216730631673061167306\\ & 116730711673071167303116730311673b,\end{aligned}$$

$$\begin{aligned}N_{40,14} = {} & 34220257621567012772325415135073344606276604575530\\ & 12775301277530127072372211175337530133753011375301\\ & 537530053753005375300537530053753b,\end{aligned}$$

$$\begin{aligned}N_{40,15} = {} & 34220257334122236772301277361327705435366604575022\\ & 12775301275160127072372211175337530133753011375301\\ & 537530053753005375300537530053753b,\end{aligned}$$

$$\begin{aligned}N_{40,16} = {} & 76741657334123012772301277330127705435362452434563\\ & 51775301273766067072372211175337530133753011375301\\ & 537530053753005375300537530053753b,\end{aligned}$$

$$\begin{aligned} N_{40,17} = {} & 360242773341230127723012773301277045373766045755301 \\ & 2774203267530127072372211175337530133312015375301 \\ & 537530053753005375300537530053753b, \end{aligned}$$

$$\begin{aligned} N_{40,18} = {} & 3422025733412301277230127733012770543536660457552 2 \\ & 3677530127756132707237221117533702213375301135160 1 \\ & 537530053753005375300537530053753b, \end{aligned}$$

$$\begin{aligned} N_{40,19} = {} & 1351602645357661152164142551760173035076217735244 6 \\ & 6622333374324370120511224672136355534775301357615 2 \\ & 020536625765402576540257654176745a, \end{aligned}$$

$$\begin{aligned} N_{40,20} = {} & 3422025252505235054455642336603603734756247644056 6 \\ & 5005405171374201202475532535655225553614216210216 1 \\ & 110723425765402576540257654056427b, \end{aligned}$$

$$\begin{aligned} N_{40,21} = {} & 4370032412742565005305171307012770543536715656364 2 \\ & 5243255120626213142142111117530256650775301053742 0 \\ & 712255421666064102164760364321350a. \end{aligned}$$

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